Enhancements for
the Lie Theory of Connected Pro-Lie Groups
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Introduction

Our book "The Lie Theory of Connected Pro-Lie Groups" appeared in March 2007. It is a research monograph, so the majority of the results were new. Obviously their proofs were new, too. Every mathematician knows that the first proof of a theorem is usually neither the most elegant nor the shortest. Also there are consequences of theorems which are not noticed until a later date. In a new theory, as is ours, there are invariably unanswered questions. Finally, and sadly there are almost always annoying typographical errors and some mathematical errors which can be corrected.

The purpose of this file called "Enhancements" is precisely that—to address all the points in the above paragraph. We begin by reproducing the Preface to our book in order to give the reader at this point an impression of its motivation and emphasis. A report on some interesting and annoying open questions follow. Subsequent sections discuss bibliographic details of more recent publications in this area and introduce the reader to some shorter proofs of key theorems due to Helge Glöckner. We conclude with Errata.

We hope that you will find this enhancement useful and that you will assist us in keeping it up-to-date by sending us additional material relevant to our book. These remarks can be sent to hofmann@mathematik.tu-darmstadt.de and/or s.morris@ballarat.edu.au

Reference

[0] Karl H. Hofmann and Sidney A. Morris, "The Lie Theory of Connected Pro-Lie Groups," European Mathematical Society Publishing House, Zürich, 2007, xv+678 pp.

Acknowledgement

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0. Preface to our book

Sophus Marius Lie (184–1899) laid the foundation of the theory named *Lie theory* in honor of its creator. Several mathematicians, likewise prominent in the history of modern mathematics, contributed to its inception in the decades following 1873, which was the year in which Lie started to occupy himself intensively in the study of what he called continuous groups, notably: Friedrich Engel, Wilhelm Killing, Élie Cartan, Henri Poincaré, and Hermann Weyl. From the beginning, however, the advance of Lie theory bifurcated into two separate major highways, which is the reason why the words *Lie Theory* mean different things to different people. Lie himself aimed at accomplishing for the solution of differential equations (in the widest sense) what Évariste Galois and Lie's countryman Niels Henrik Abel achieved for the solution of algebraic equations: A profound understanding and, to the best extent possible, a classification in terms of groups. Even though Lie considered himself a "geometer," he created a territory of analysis that is called "Lie Theory" by those working in it, and that is represented by the well-known text by Peter J. Olver entitled "Applications of Lie Groups to Differential Equations" [Springer-Verlag, Berlin, New York, etc., 1986]. We should say in the beginning, that the project of Lie theory which we shall discuss in this book, in philosophy and thrust, does not belong to this line.

A second highway was taken by Killing and Cartan. It led to a study of what soon became known as Lie algebras, of the group and structure theory of Lie groups, and to the geometry of homogeneous spaces. The latter notably yielded the classification of symmetric spaces by Élie Cartan. At long last it merged into the encyclopedic attempt by Nicolas Bourbaki of the nineteen hundred sixties and seventies, to summarize what had been achieved, and to the emergence of an immense collection of textbooks at all levels. In 1976 Jean Dieudonné quipped *"Les groupes de Lie sont devenus le centre de mathématique. On ne peut rien faire de sérieux sans eux.* (Lie groups have moved to the center of mathematics. One cannot seriously undertake anything without them.) By and large, in this line of "Lie theory" the words meant the structure theory of Lie algebras and Lie groups, and in particular how the latter is based on the former.

The term 'Lie group' originally meant 'finite-dimensional Lie group' and most people understand the words in this sense today. However even Sophus Lie spoke of "unendliche Gruppen" by which he meant something like infinite-dimensional Lie groups. But reasonable concepts of dimension were not yet available in the 19th century before topology was on its way. And indeed Lie's attempts in this direction did not appear to have gotten off the ground.

The significance of Lie's discoveries was emphasized by David Hilbert by raising the question in 1900 whether (in later terminology) a locally euclidean topological group is in fact an analytic group in the sense of Lie. This was the fifth of his famous 23 problems which foreshadowed so much of the mathematical creativity of the 20th century. It required half a century of effort on the part of several generations of eminent mathematicians until it was settled in the affirmative. Partial solutions came along as the structure of topological groups was understood better and better: Hermann Weyl and his student F. Peter in 1923 laid the foundations of the representation and structure theory of compact groups, and a positive answer to Hilbert's Fifth Problem for compact groups was a consequence, drawn by John von Neumann in 1932. Lev Semyonovich Pontryagin and Egbert Rudolf van Kampen developed in 1932, respectively, 1936, the duality theory of locally compact abelian

groups laying the foundations for an abstract harmonic analysis flourishing throughout the second half of the 20th century and providing the central method for attacking the structure theory of compact abelian groups via duality. Again a positive response to Hilbert's question for locally euclidean abelian groups followed in the wash.
One of the most significant and seminal papers in topological group theory was published in 1949 by Kenkichi Iwasawa, some three years before Hilbert's Problem was finally settled by the concerted contribution of Andrew Mattei Gleason, Dean Montgomery, Leon Zippin, and Hidehiko Yamabe. It was Iwasawa who clearly recognized for the first time that the structure theory of locally compact groups reduced to that of compact groups and finite-dimensional Lie groups *provided* one knew that they happen to be approximated by finite-dimensional Lie groups in the sense of projective limits, in other words, if they were pro-Lie groups in our parlance. And this is what Yamabe established in 1953 for all locally compact groups which have a compact factor group modulo their identity component—almost connected locally compact groups as we shall say. The most influential monograph collecting these results was the book by Montgomery and Zippin of 1955 with the title "Topological Transformation Groups". The theories of compact groups and of abelian locally compact groups had introduced in the first half of the century classes of groups with an explicit structure theory without the restriction of finite-dimensionality, and in the middle of the century these results opened up an explicit development for numerous results on the structure theory of locally compact groups.

What are the coordinates of our book in this historical thread?
It was recognized in 1957 by Richard Kenneth Lashof that any locally compact group G has a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is appropriately defined, then the exponential function $\exp\colon \mathfrak{g} \to G$ is supplied along with the definition. Yet the fact that these observations are the nucleus of a complete and rich, although infinite-dimensional Lie theory was never exploited. The present book is devoted to the foundations, and the exploitation of such a Lie theory. At a point in the overall historical development where infinite-dimensional Lie theories gain increasing acceptance and attract much interest, this appears to be timely. The Lie theory we unfold is based on projective limits, both on the group level and on the Lie algebra level. We shall find it very helpful that category theory, as a tool for the "working mathematician" as Saunders MacLane formulated it, is so well developed that we see immediately what we need, and we shall exploit it. In our case, we need the theory of limits in a complete category, that is, in a category in which all limits exist, and we need the theory of pairs of adjoint functors, which is closely linked with limits.

The machinery of projective limits is familiar to mathematicians dealing with profinite groups in their work on Galois theory and arithmetics, quite generally. But the apparatus of projective limits is also familiar to mathematicians dealing with compact groups, their representation theory and abstract harmonic analysis. Indeed all group theoreticians working on the structure theory of locally compact groups encounter projective limits sooner or later. In this book we shall call projective limits of projective systems (or, as some authors say, inverse systems) of finite-dimensional Lie groups. That is, pro-Lie groups relate to finite-dimensional Lie groups exactly as profinite groups relate to finite groups.
However, in the theory of locally compact groups, one encounters a special kind of projective limit, namely, limit situations where limit maps and bonding maps are, that is, are closed

continuous homomorphisms between locally compact groups having compact kernels. Some authors call such maps perfect. This type of projective limit has a significant element of compactness already built into its definition, and it is this type of limit that has shaped the intuitions of group theoreticians for fifty years or more.

From the vantage point of category theory, however, such a restriction is entirely unnatural, as is indeed the entire focus on locally compact pro-Lie groups: The class of locally compact groups is not even closed under the formation of products—as the example of the groups $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{Z}^{\mathbb{N}}$ shows immediately. Mathematicians will be naturally attracted to the problem of eliminating the focus on locally compact groups. As one proceeds in the direction of pro-Lie groups in general, however, one comes to realize that the restriction to locally compact groups is unnatural also for reasons that are entirely interior to the mathematics of topological groups and Lie groups. For several years we have been engaged in the laying of the foundations of a general theory of the category of pro-Lie groups. The results are presented in this book. On the first 60 pages, the reader will find a panoramic overview of what is contained in its 14 chapters, and the user of the book should get a more compact overview by perusing its table of contents.

The Lie theory of finite-dimensional Lie groups works because for a *connected* Lie group G, its Lie algebra $\mathfrak{g}$ and its exponential function $\exp\colon \mathfrak{g} \to G$ largely determine the structure of G. We hasten to add that, except for the case that G is simply connected, they do not do so completely. As the title of our book indicates, we focus on a Lie theory for connected pro-Lie groups. As a consequence, our structure theory is one that is mainly concerned with connected pro-Lie groups, sometimes going a bit further, but rarely much beyond almost connected groups. In view of Yamabe's Theorem, the structure theory of connected or almost connected pro-Lie groups applies at once to connected or even almost connected locally compact groups.

There are several key elements to the structure theory of pro-Lie groups.

Firstly, a thorough understanding of the working of projective limits is needed without the crutch of thinking in terms of proper maps all the time. Chapter 1 deals with many facets of this issue. But only after Chapter 3 will we have understood all aspects of what this means for the very definition of pro-Lie groups itself.

Secondly, the entire theory depends on our accepting that pro-Lie groups, even though not being Lie groups, nevertheless have a working Lie theory, complete with the appropriate Lie algebras which we shall call *pro-Lie algebras* and working exponential functions that mediate between pro-Lie groups and their Lie algebras. Indeed we must become aware at an early stage that there is a good Lie algebra functor from the category of pro-Lie groups to the category of pro-Lie algebras. One of the very positive side effects of facing wider categories than the conventional ones in developing a Lie theory is that this enlargement of scope forces us to realize in great clarity that the Lie algebra functor is opposed by a Lie group functor that encapsulates lucidly the contents of Lie's Third Fundamental Theorem. This applies to the classical situation as well, but it is not recognized there because the theory of universal covering Lie groups, while providing topologically satisfying results in general, tends to obscure the precise functorial set-up. Since for pro-Lie groups a classical covering theory is impossible as, one knows from the theory of compact connected abelian groups, it is mandatory that one understands the functorial background of a more general universal covering theory. We shall discuss this in Chapters 2, 4, 6 and 8.

Thirdly, the success of the structure theory of pro-Lie groups depends in a large measure on our success in dealing with the structure theory of pro-Lie algebras. This pervades the whole book, but most of this is done in our rather long Chapter 7. The point is that the topological vector spaces underlying pro-Lie algebras are what we call weakly complete topological vector spaces, because they are exactly the duals of real vector spaces given the weak $*$-topology, that is, the topology of pointwise convergence of linear functionals. Since the vector space duality is crucial for this class of topological vector spaces and hence for the structure theory of pro-Lie algebras we present the essential features of the linear algebra of weakly complete topological vector spaces in an appendix, namely, Appendix 2. The relevance of weakly complete topological vector spaces in the structure theory of pro-Lie groups themselves is evidenced in that chapter in which we discuss the structure of commutative pro-Lie groups, and that is Chapter 5.

With all of these foundations done, the Lie and structure theory of pro-Lie groups can proceed, as it does in Chapters 9 through 13. This preface is not the place to go into the details, but we shall present to our readers in the beginning of the book, in our panoramic overview, the results which we obtain. One of the lead motives of our structure theory is to reduce the structure of connected pro-Lie groups in the optimal extent possible to the structure theory of compact connected groups, weakly complete topological vector spaces, and finite-dimensional Lie groups. We will prove some major structure theorems which expose that we, in essence, achieve this goal.

1. Open problems on pro-Lie groups

References

[1.1] Glöckner, H., *Simplified Proofs for the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma*, J. of Lie Theory **17** (2007), 899–902.

[1.2] Hofmann, K. H., and S. A. Morris, *On the Pro-Lie Group Theorem and the Closed Subgroup Theorem*, submitted 8 pp.

[1.3] —, *Contributions to the Structure Theory of Connected Pro-Lie Groups*, submitted 9 pp.

[1.4] Lee, D. H., *Supplements for the identity component in locally compact groups*, Math. Z. **104** (1968), 28–49.

[1.5] George Michael, A. A., *On Inverse Limits of Finite-Dimensional Lie Groups*, J. Lie Theory **16** (2006), 221–224.

The following list of open questions is grouped into subject matter within the theory of pro-Lie groups. Various comments are intended as a guidance.

1. Foundations of Pro-Lie Groups

Despite the amount of information provided in Chapter 3 of our book [0] the papers [1.1–1.4] above and the additional material presented in Sections 2 and 3 below, a definitive answer to the following question is not known:

Problem 1.1.1. Let G be a pro-Lie group and G_0 the connected component of the identity. *Is the component factor group G/G_0 complete?*

In [0], 3.31, p. 154 it was shown that the open normal subgroups of G/G_0 form a basis of the filter of identity neighborhoods. In [0], Theorem 4.1 it was shown that the property of having arbitrarily small normal subgroups N such that G/N is a Lie group passes down to quotient groups. In [0], Corollary 4.11 we saw that there are indeed pro-Lie groups with incomplete quotient groups. There are sufficient conditions on G_0 for Problem 1.1.1 to have an affirmative answer: (i) G_0 is locally compact or is isomorphic to a product of first countable groups (see [0] Theorem 4.28 and subsequent discussion on p. 204), (ii) G_0 is abelian (first factor the maximal compact connected subgroup C, notethat $(G/C)_0 \cong \mathbb{R}^J$ by [0], Theorem 5.20, then apply (i). (iii) G_0 is simply connected reductive (see [0], p. 204). The significance of an affirmative answer to Problem 1.1.1 is that [0], Theorem 4.28(i) would simply read *the quotient group of a pro-Lie group modulo an almost connected closed normal subgroup is a pro-Lie group.*

A basic ingredient of the foundations of pro-Lie group theory is the set $\mathcal{N}(G)$ of normal subgroups such that G/N is a Lie group. Indeed if G is a complete group and $\mathcal{N}(G)$ has arbitrarily small elements, then G is a pro-Lie group. The good news about $\mathcal{N}(G)$ is that it is completely canonically attached to a topological group G, containing at least G itself. It contains $\{1\}$ iff G is a Lie group. The bad news is that in many cases, even the simplest ones, $\mathcal{N}(G)$ is much too large for most intents and purposes. If G is a Lie group, then $\mathcal{N}(G)$ contains all closed normal subgroups; If G is an abelian Lie group, $\mathcal{N}(G)$ is the set of all closed subgroups. For $G = \mathbb{R}$ or $G = \mathbb{Z}$, this means that $\mathcal{N}(G)$ contains all subgroups of the form $r{\cdot}\mathbb{Z}$, $r \in G$ while it would be good enough consider the set of the singleton subgroup if G is a Lie group itself. If G is a simply connected pro-Lie group, the set $\mathcal{N}_0(G)$ of all identity components N_0 of the members N of $\mathcal{N}(G)$ is a filter basis that is cofinal in $\mathcal{N}(G)$.

In [0], Chapter 6 in Definition 6.1 on p. 249 we introduced the filter basis $\mathcal{NS}(G)$ of closed normal subgroups such that G/N is simply connected.

In our discussion of the completeness of quotients beginning with 4.24 on p. 195 we started a process of "thinning out" $\mathcal{N}(G)$, deriving the subset $\mathcal{M}(G)$ which, in [0], 4.25 we found to be cofinal in $\mathcal{N}(G)$ if G is almost connected. More on this is to be found in [0], Corollary 9.45. In Definition 4.26 we use the set $\mathcal{M}(G)$ to define what we call the Z-topology, a tool we use in showing the at least certain quotients of pro-Lie groups are again pro-Lie groups. There is a need for consolidation of a theory of various relevant subsets of $\mathcal{N}(G)$; the information we have is currently dispersed over various parts of [0] even though the issue as such belongs to the foundations of the theory of pro-Lie groups.

Problem 1.1.2. Let G be a pro-Lie group. *Is there a coherent and systematic theory of the sets $\mathcal{N}(G)$, $\mathcal{NS}(G)$, $\mathcal{M}(G)$, $\mathcal{N}_0(G)$. Particular emphasis is to be given the case that G is almost connected, that is, that G/G_0 is compact.*

2. Compact Generation

A topological group G is *compactly generated* if there is a compact subset $C \subseteq G$ such that $G = \langle C \rangle$, that is, that G is *algebraically* generated by C.
In [1.3] above we have the following result, arising in the context of [0], Corollary 12.87.

Proposition. *Let G be a connected pro-Lie group and C one of its maximal compact subgroups. Then the following statements are equivalent:*

(i) *G is compactly generated.*
(ii) *G is σ-compact.*
(iii) *G/C is σ-compact.*
(iv) *G/C is locally compact.*
(v) $\operatorname{rank}(G) = \dim G/C < \infty$.
(vi) *G is locally compact.*

This suggests the following question:

Problem 1.2.1 *Is a compactly generated pro-Lie group locally compact?*

For connected pro-Lie groups, the preceding Proposition gives an affirmative answer. In [0], Chapter 5, Theorem 5.20(v), p. 230, it was shown that in every abelian pro-Lie group G, the identity component G_0 and the union of all compact subgroups generate a closed characteristic subgroup G_1 and that G/G_1 is a pro-discrete group without nontrivial compact subgroups. Theorem 5.32 proves that *in a compactly generated abelian pro-Lie group G, the subgroup G_1 is locally compact and G/G_1 is a compactly generated prodiscrete abelian group without compact nonsingleton subgroups.*
Thus even the following question has not been answered.

Problem 1.2.1.1. *Is an abelian prodiscrete compactly generated group without nondegenerate compact subgroups discrete?*

If the answer is yes here then it would follow that a compactly generated abelian pro-Lie group G is locally compact and then would be isomorphic to $\mathbb{R}^n \times C \times \mathbb{Z}^n$ for the unique maximal compact subgroup C of G.

3. Abelian Pro-Lie Groups

Despite a full chapter on the structure theory of abelian pro-Lie group presented in [0], such a theory is far from complete. Partial results are found in [0] from 5.33, p. 238 to 5.41 on p. 241, followed by additional structural result until 5.47 on p. 245.
Therefore we formulate:

Problem 1.3.1. *Is there a convincing structure and character theory of prodiscrete abelian groups?*

Problem 1.3.1.1. Treat the special case of *compact-free prodiscrete groups.* As a typical example, the kernel $F = \operatorname{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ of the morphism is a closed and nondiscrete subgroup of $\operatorname{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{\mathbb{R}}$ and is algebraically isomorphic to $\mathbb{Z}^{(\mathbb{N})}$. *Investigate the character group and bicharacter group of F.* (See [0], pp. 173ff.)

Problem 1.3.1.2. Treat the special case of *prodiscrete groups consisting of compact elements.*

A relevant example is as follows: Let $\mathbb{Z}(2) = \mathbb{Z}/2\mathbb{Z}$ be the group of 2 elements. The group $\mathbb{Z}(2)^{(\aleph_1)}$ has a nondiscrete group topology making it a prodiscrete (hence pro-Lie) group T. This is a torsion group, hence T is the union of all compact subgroups. The bidual $\widehat{\widehat{T}}$ is discrete; the evaluation morphism $T \to \widehat{\widehat{T}}$ is bijective and open but not continuous. (See [0], Example 14.15)

4. Topological Splitting of Maximal Compact Subgroups

In [0], 12.81–12.86 we proved the following result:

Theorem. *Let G be a connected pro-Lie group. Then there is a closed subset $M \subseteq G$ and a compact subgroup $C \subseteq G$ such that*

(i) *there is a homeomorphism $\phi: \mathbb{R}^J \to M$ for a set J.*
(ii) *Every compact subgroup has a conjugate contained in C.*
(iii) *$(v, c) \mapsto \phi(v)c: \mathbb{R}^J \times C \to G$ is a homeomorphism.*

This theorem has formidable consequences some of which are discussed in [0] and in [1.3]. For instance, together with Theorem 6 of [5.2] in Section 5 below it implies that connected pro-Lie groups are Baire spaces. It is reasonable to expect that the answer to the following question is positive.

Problem 4.1. *Does the theorem above remain valid if G is an almost connected pro-Lie group?*

Similarly, it is not unreasonable that the answer to the following question is yes:

Problem 4.2. Let G be an almost connected pro-Lie group. *Does there exist a profinite subgroup P such that $G = G_0P$?*

The answer is yes if G is locally compact G (see [1.4]). The authors shall deal with Problems 4.1 and 4.2 in a forthcoming publication.

2. The Closed Subgroup Theorem and Glöckner's Shorter Alternative Proof of the Pro-Lie Group Theorem

References

[2.1] Glöckner, H., *Simplified Proofs for the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma*, J. of Lie Theory **17** (2007), 899–902.

[2.2] Hofmann, K. H., and S. A. Morris, *On the Pro-Lie Group Theorem and the Closed Subgroup Theorem*, submitted 8 pp.

[2.3] George Michael, A. A., *On Inverse Limits of Finite-Dimensional Lie Groups*, J. Lie Theory **16** (2006), 221–224.

Recall that a *pro-Lie group* is a complete topological group G in which every identity neighborhood contains a normal subgroup N such that G/N is a Lie group ([0], Definition 3.25, p. 149). The Pro-Lie Group Theorem is fundamental and states ([0], Theorem 3.34, p. 157)

2.1. *Every projective limit of Lie groups is a pro-Lie group.*

An alternative (but not self-contained) proof was given by Adel George Michael in [2.3]). A proof that is very much in the spirit of our book [0] was recently given by Helge Göckner. It takes off at the point of p. 148 of [0] from the end of the section called *Weakly Complete Topological Vector Spaces and Lie Algebras* (pp. 143–148) and from there provides in [2.1] a very elegant and short proof of 2.1 above. As a second very valuable observation he provides the insight, that in this very section provides the basis for a very short proof of the One Parameter Subgroup Lifting Lemma ([0], Lemma 4.19, p. 184; see also [0], Definition 2.6 on p. 110 and Theorem 2.19 on p. 120):

2.2. *Let $f\colon G \to H$ be a quotient morphism of topological groups and assume that G is a pro-Lie group. Then $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(H)$ is surjective.*

In Chapter 4 of our book [0], the consequences of this result are discussed in 4.20ff., pp. 188ff. The significance of Glöckner's observation regarding the One-Parameter Subgroup Lifting Lemma is that it obviates the rather lengthy proof of [0], Lemma 4.19, p. 184ff.

Another good use of the very same section (pp. 143–148) was made by us in [2.2], where we prove the following result:

2.3. (a) *Let H be an almost connected closed subgroup of a pro-Lie group G and let $M \in \mathcal{N}(G)$. Then there is a closed normal subgroup N of $\overline{HM}$ such that $N \subseteq M$ and the standard bijection*

$$f_N\colon H/(H \cap N) \to HN/N, \quad f_N(h(H \cap N)) = hN,$$

is an isomorphism of Lie groups.
(b) *If H is normal in G, then N is constructed to be normal in G, that is, $N \in \mathcal{N}(G)$.*

Theorem 2.3 above secures another instance of the validity of the so-called Second Isomorphy Theorem $H/(H \cap N) \cong HN/N$ in the category of topological groups. Usually, if it is true at all in the sense that the isomorphism holds algebraically *and topologically*, some application of the Open Mapping Theorem is involved (see e.g. [0], Corollary 9.62, p. 413) which is not the case in this instance. Theorem 2.3 pertains to the theory of projective limits. Normally, hypotheses on the projective system lead to conclusions on the limit. It is rare that, as in Theorem 2.3, assumptions on the limit entail conclusions on the projective system.

In the general context of our book [0], the significance of Theorem 2.3 lies in its pertaining to the so-called "Closed Subgroup Theorems". Our book contains one such theorem on topological groups which are given as projective limits in the category of complete topological groups (see [0], Theorem 1.34, p. 96f.), and it contains another one for the category of pro-Lie groups (see [0], Theorem 3.35, p. 158). We need to go into some detail at this point.
In the course of his study [2.1], Helge Glöckner discovered that in our Closed Subgroup Theorem [0], 1.34, the statements (iii), and (iv) on p. 97 are false and need to be withdrawn (see Section "Errata" below). Statement (iii) was never used with the exception of deriving (iv), but (iv) was used in a number of places in the book and these have to be discussed. It is here where Theorem 2.3 above comes in as a partial replacement of statement [0] 1.34(iv). It is partial for two reasons: Firstly, in 1.34(iv), the isomorphism $H/(H \cap N) \cong HN/N$ was asserted for all $N \in \mathcal{N}(G)$ whereas in 2.3 it was proved only for a cofinal subset of subgroups N. Secondly, in 1.34, statement (iv) was asserted for all complete G which are projective limits of complete quotients G/N, while 2.3 above is asserted for pro-Lie groups G only. The full analogy between the abandoned statement 1.34(iv) and Theorem 2.3 above is attained for *normal* subgroups H.
In as much as Theorem 2.3 above corrects [0], Theorem 1.34 (iii,iv), it cannot be inserted in the vicinity of [0], 1.34, since it deals with pro-Lie groups and is based on material introduced in [0], Chapter 3. The precise coordinates involved in dealing with the rectification of the omission of 1.34 (iii,iv) are described in [2.2] and in the List of Errata below.

3. Glöckner's and Neeb's Locally Convex Lie Groups versus Pro-Lie Groups

References
[3.1] Glöckner, H., and K.-H. Neeb, "Infinite-Dimensional Lie Groups," book in preparation.
[3.2] Hofmann, K. H., and K.-H. Neeb, *Pro-Lie groups which are infinite-dimensional Lie groups*, Math. Proc. Cambridge Phil. Soc. (2008), to appear.
[3.3] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jap. J. Math., **1** (2006), 291–468.

Calculus on manifolds which are modeled on Banach spaces is rather well known and thus one is reasonably familiar with Lie groups having an identity neighborhood that is diffeomeorphic to an open subset of a Banach space. Doing calculus on manifolds which are modelled on locally convex topological vector spaces is considerably harder, and, accordingly, a smooth Lie group theory for Lie groups on such *locally convex manifolds* is much harder to come by; in short, one refers to a group G as a *locally convex Lie group* if it supports a group structure in the category of smooth locally convex manifolds. A comprehensive source will become available in the form of the multivolume work [3.1]. Meanwhile, Karl-Hermann Neeb's booksize survey paper [3.3] is an excellent introduction.
In our book [0], we provide a Lie theory for the class of pro-Lie groups in terms of topololgical groups and without reference to calculus. It is, therefore, a natural question, whether there are pro-Lie groups that are locally convex Lie groups in the sense of Glöckner and Neeb, and indeed if that is the case, how they are to be characterized and whether the pro-Lie group Lie theory (that is, associated Lie algebras and exponential functions) agree with the smooth Lie theory attached to a locally convex Lie group via smooth calculus.
This is where reference [3.2] above comes in and answers all of these questions.
Let us record here some of the most crucial results which connect the Lie theory of pro-Lie groups and smooth Lie theory of locally convex Lie groups.

We shall call a topological group G *locally contractible* if G has an identity neighborhood U which is contractible to a point in G, that is, there is a homotopy $F\colon [0,1]\times U \to G$ such that $F(0,-)$ is the inclusion $U \to G$ and $F(1,-)$ is the constant function $U \to G$ taking the value 1. Of course, a group is locally contractible if it has a contractible identity neighborhood.

Theorem 3.1. *A pro-Lie group G carries a locally convex Lie group structure compatible with its topology if and only if it is locally contractible.*

Let us call a pro-Lie algebra *smooth*, if it occurs as the Lie algebra of some locally contractible pro-Lie group. The following theorem now classifies smooth pro-Lie algebras: But first recall from [0], Corollary 7.29 on p. 283, Theorem 7.48, p. 292, Theorem 7.52, p. 7.52 that every pro-Lie algebra $\mathfrak{g}$ is a semidirect algebraic and topological sum of a prosolvable radical $\mathfrak{r}(\mathfrak{g})$ and a product $\prod_{j\in J}\mathfrak{s}_j$ for a family of finite dimensional simple real Lie algebras $\mathfrak{s}_j$. The universal simply connected pro-Lie group $\Gamma(\mathfrak{g})$ attached to a pro-Lie algebra was constructed in [0] in Chapters 2,6, and 8. Recall that there is a categorical equivalence between the categories of pro-Lie algebras and simply connected pro-Lie groups via $\mathfrak{g}\mapsto\mathfrak{L}(\mathfrak{g})$ and $G\mapsto\mathfrak{L}(G)$. A compact connected group G is *nearly abelian* if its commutator group $[G,G]$ is finite dimensional.

Theorem 3.2. *For a pro-Lie algebra $\mathfrak{g}$, the following are equivalent:*
(1) *$\mathfrak{g}$ is the Lie algebra of a locally convex Lie group G with smooth exponential function.*
(2) *$\mathfrak{g}$ has a Levi decomposition $\mathfrak{g} \cong \mathfrak{r}(\mathfrak{g}) \rtimes \mathfrak{s}$, where only finitely many factors in $\mathfrak{s} \cong \prod_{j\in J} \mathfrak{s}_j$ are not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.*
(3) *The corresponding simply connected universal group $\Gamma(\mathfrak{g})$ is locally contractible.*
(4) *The maximal compact subgroups of $\Gamma(\mathfrak{g})$ are nearly abelian.*
(5) *There exists a locally contractible pro-Lie group G with $\mathfrak{L}(G) \cong \mathfrak{g}$.*

In locally convex smooth Lie group theory, the issue of the existence of a smooth exponential function is by no means trivial. A Lie group G is called *regular* if for each smooth curve $\xi\colon [0,1] \to \mathfrak{g}$, the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = \gamma(t)\cdot\xi(t) = T_{\mathbf{1}}(\lambda_{\gamma(t)})\cdot\xi(t)$$

has a smooth solution $\gamma_\xi\colon [0,1] \to G$, and the map

$$\mathrm{ev}\colon C^\infty([0,1],\mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth. Then $\exp_G(x) := \mathrm{ev}(x)$, where $x \in \mathfrak{g}$ is identified with a constant function $[0,1] \to \mathfrak{g}$, yields an exponential function of G.
In the context of smooth pro-Lie groups we do get results like the following:

Theorem 3.3. *For a pro-Lie algebra $\mathfrak{g}$, the following assertions hold:*
(1) *If G is a Lie group with a smooth exponential function and $\mathfrak{L}(G) = \mathfrak{g}$, then $\mathfrak{g}$ is smooth.*
(2) *If $\mathfrak{g}$ is smooth, then there exists a unique simply connected connected regular Lie group, which is isomorphic to $\Gamma(\mathfrak{g})$ as a topological group.*
(3) *If G is any connected regular Lie group for which $\mathfrak{g} = \mathfrak{L}(G)$, then G is a quotient of $\Gamma(\mathfrak{g})$ by a discrete central subgroup. A subgroup $D \subseteq Z(\Gamma(\mathfrak{g}))$ is discrete if and only if it is finitely generated and its intersection with the identity component $Z(\Gamma(\mathfrak{g}))_0 \cong \mathfrak{z}(\mathfrak{g})$ is discrete.*

A Lie group G is called *locally exponential* if it has a smooth exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ mapping some open 0-neighborhood in $\mathfrak{L}(G)$ diffeomorphically onto an open identity neighborhood in G.
A pro-Lie algebra $\mathfrak{g}$ is locally exponential if and only if the set of exp-regular points, that is, the set of all $x \in \mathfrak{g}$ for which $\mathrm{Spec}(\mathrm{ad}\,x) \cap 2\pi i\mathbb{Z} = \{0\}$, is a 0-neighborhood. Now we are ready for

Theorem 3.4. *A pro-Lie group G is locally exponential if and only if it is locally contractible and $\mathfrak{L}(G)$ is a locally exponential Lie algebra. A pro-Lie algebra $\mathfrak{g}$ is locally exponential if and only if the set of* exp*-regular points, that is, the set of all $x \in \mathfrak{g}$ for which*

$$\mathrm{Spec}(\mathrm{ad}\,x) \cap 2\pi i\mathbb{Z} = \{0\},$$

is a 0-neighborhood.

For additional details the reader should consult reference [3.2] above.

4. Errata

In counting lines, headlines do not count.

page 11,	line 7	read: "the underlying topological vector space".
page 13,	line 7	read: "Hom($\widehat{G}, \mathbb{R}$),"
page 22,	line 17	read: "card $\alpha \le$ card$\overline{\mathfrak{g}}$" via"
page 22,	line -16	read: "$\overline{\gamma} \le \omega$, then"
page 23,	line -1	read :"$\cdots \cap \mathfrak{r}(\mathfrak{g}) = \cdots$"
page 25,	line 16	replace "an inner" by "a special"
page 26,	line 1	read: "denote its nilradical or"
page 28,	line 2	read: "of pro-Lie algebras translates"
page 28,	line 3	read: "to simply connected pro-Lie groups."
page 28,	line -17	read: "immersed connected submanifold."
page 29:	line 6	read: if and only if there"
page 30,	line 4	read: bijection onto the set of closed subalgebras of $\mathfrak{L}(G)$ with
page 33,	line 7of Theorem 42	replace $\ker(\rho_1)_N$ by $\ker(\rho)$.
page 36,	line 6	drop the word "Lie".
page 36,	line 10	delete "$\mathfrak{g} = \mathfrak{L}(G)$ and".
page 36,	lines 11 and -14	replace "card $\mathfrak{g}$" by card $\overline{G}$.
page 36,	line 18	read: "$\overline{\gamma} \le \omega$" instead of "$\gamma \le \omega$".
page 38,	line 2	delete "transfinitely"
page 41,	line -4	read: "$N \rtimes_\iota G$" not "$G \rtimes_\iota N$".
page 42,	line 2 of Theorem 63	replace "semisimple" by "reductive".
page 42,	line 3 of Theorem 63	replace "$\langle \exp_G \mathfrak{g} \rangle$" by "$\langle \exp_G \mathfrak{s} \rangle$".
page 43,	line 7	replace "$z \in Z$" by "$z \in Z(G)_0$".
page 46,	part (iv) of Theorem 72	omit.
page 46,	line 2 of Theorem 73	replace "semisimple" by "reductive".
page 46,	line-9	read "$\mathfrak{g}$-module as".
page 47,	line 1:	read "$x \in L$" instead of "$x \in \mathfrak{g}$".
page 48,	Part (v) of Theorem 76	delete Part (H) and rename (I) into (H).
page 48,	line -15	read $\mathfrak{z}(\mathfrak{g}) + \mathfrak{k}$ without overbar (cf. A2.12(c)!)
page 52,	line -20	read $\exp_{\Gamma(\mathfrak{g})} \mathfrak{g}$.
page 54,	line -2	replace "maximal potentially compact connected" by "maximal compactly embbedded connected".
page 55,	line -17	replace "zc" by "vzc".
page 56,	line 17	delete "Lie".
page 59,	line -5	read "group G is homeomorphic".
page 61,	lines -9,-5	replace "$\mathcal{N}(G)$ by "nilcore(G)".
page 67,	line 2	read "and $q = \mathrm{pr}_B\, e$."
page 70,	line 3	delete the word "full".
page 72,	line 12	delete "that".
page 89,	line 9	the north-east corner of the diagram should read $\lim_{P \in \mathcal{N}} G/P$
page 89,	line 1 of 1.27(iv)	read "be the morphisms induced".
page 89,	lines -14to-10	remove.

page 90,	line 3	insert comma before "i.e.," and replace "i.e" by "that is".
page 90,	line 8	read: there is a $k \in J$, $j \leq k$, and
page 90,	lines 21,22,23	omit.
page 90,	line -5	after "onto its image" insert:

", and so, while the G_j may come from a desirable class, such as the class of Lie groups, the quotients G/N may not be in that class"

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page 94,	line 1 of Theorem 1.30	replace "filter" by "filter basis".
page 96,	line 2 of 1.33(ii)	omit "and η is an isomorphism inverting γ_G.
page 96,	lines -2,-1 of the proof of 1.33	omit "It is straightforward. . .$\eta^{-1} = \gamma_G$."
page 96,	after the proof of 1.33	insert the following paragraph:

It is straightforward that $\eta\gamma_G = \mathrm{id}_G$, and since γ_G is an isomorphism by Theorem 1.33(i), this shows once more that $\eta^{-1} = \gamma_G$ as we proved in Theorem 1.27(ii) without completeness hypothesis.

---

page 97	lines 1–4	replace by the following:

(iii) The limit maps $\mu_M\colon \lim_{N\in\mathcal{N}} HN/N \to HM/M$, $M \in \mathcal{N}$, are surjective morphisms.
(iv) The standard morphisms $H/(H \cap N) \to HN/N$ are bijective morphisms of topological groups.

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page 99,	line 3	The assertion that μ_{MN} is a quotient morphism is false. Therefore, conclusions (iii) and (iv) of the Theorem had to be replaced.
page 98,	from line -2	until
page 99,	line 11	not counting the diagram, replace by

(iii) We must show that the limit morphisms $\mu_M\colon \lim_{N\in\mathcal{N}} HN/N \to HM/M$ are surjective: An element ξ of HM/M is a coset kM for some $k \in H$. After (ii) we know that $h \mapsto (hN)_{N\in\mathcal{N}} : H \to \lim_{N\in\mathcal{N}} HN/N$ is an isomorphism. Then $\mu_M(k) = kM = \xi$.
(iv) We recall that

$$\eta\colon H/(H \cap N) \to HN/N,\ \eta\big(h(H \cap N)\big) = hN,$$

is algebraically an isomorphism and that it is the unique morphism of topological groups induced by the continuous morphism $h \mapsto hN : H \to HN/N$ whose kernel is $H \cap N$.

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page 99,	line -13	replace "G_a" by "G_α".
page 127,	line 2	read "$\pi_A\colon F(U(A)) \to A$ such that".

page 136, lines -6, -5 reformulate as follows:

It follows immediately from Proposition 3.2 that LieProGr is also closed under passing to closed subgroups. In Theorem 3.35 this will be corroborated in a different fashion.

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page 139, line -4	Deline entire line.
page 139, line -3	read "Let L be a complete topological"
page 140, line 3	After "of ideals" insert "converging to 0 and"
page 143, line 5 of 3.18	read $\gamma_{jk_j}(\mathfrak{g}_{k_j}) \subseteq \gamma_j(\mathfrak{g})$.
page 144, line 13	read "...Limits 1.27(iv) to...".
page 144, line 15	read (Appendix 2, Theorem A2.12(b)).
page 144, line -1	line starts with "$\exp \mathfrak{a}_j$", not with "$\mathfrak{a}_j$".
page 146, line -5	line starts with "$\mathfrak{L}(G_0)$", not with "$\mathfrak{L}(B)$".
page 147, bottom display diagrams	read rightmost down arrow:

$$\begin{array}{c} H \\ \big\downarrow {\scriptstyle \mathrm{inc}_A \circ \varepsilon} \\ G^0 \end{array}$$

---

page 148, line 1 read:

$$\beta_j \stackrel{\mathrm{def}}{=} \overline{f}_{jk_j} \circ f_{k_j}|G^0 : G^0 \to H_j$$

---

page 148, line 4	read $H = \lim_{j\in J} H_j$
page 149, line 3 of 3.25	read "normal subgroup N such that".
page 151, line -3	Replace "**Example 3.28.** All" by **Exercise E3.7.** Show that all
page 151, line -2	From "This is" to the end of the page, read:

[Hint. This is an easy exercise from Definition 3.25 of a pro-Lie group and the discussion of 3.4 in Exercise E3.2.]

---

page 151, line -1 At the bottom of the page insert the following text:

We now prove a result which belongs to the context of the Closed Subgroup Theorem 1.34.

Theorem 3.28. (a) *Let H be an almost connected closed subgroup of a pro-Lie group G and let $M \in \mathcal{N}(G)$. Then there is a closed normal subgroup N of $\overline{HM}$ such that $N \subseteq M$ and* *the standard bijection*

$$f_N\colon H/(H\cap N)\to HN/N. \quad f_N(h(H\cap N))=hN,$$

is an isomorphism of Lie groups.
(b) *If H is normal in G, then N is constructed to be normal in G, that is, $N\in\mathcal{N}(G)$.*

We shall prove this theorem in several steps through a sequence of lemmas. We are given the pro-Lie group G with its filter base $\mathcal{N}(G)$ of closed normal subgroups N such that G/N is a Lie group. Then G may be identified with the projective limit of the system

$$\{p_{MN}\colon G/N\to G/M : N\subseteq M,\ m,n\in\mathcal{N}(G)\}.$$

By Theorem 1.34, a closed subgroup H of G gives rise to three projective systems of topological groups:

$$\begin{aligned}
&\{q_{MN}\colon \overline{HN}/N\to\overline{HM}/M : N\subseteq M,\ M,N\in\mathcal{N}(G)\},\\
&\{r_{MN}\colon HN/N\to HM/M : N\subseteq M,\ M,N\in\mathcal{N}(G)\},\\
&\{s_{MN}\colon H/(H\cap N)\to H/(H\cap M) : N\subseteq M,\ M,N\in\mathcal{N}(G)\},
\end{aligned}$$

and all of them have H as limit by Theorem 1.34, as is illustrated in the following diagram:

$$\begin{array}{ccccc}
\frac{H}{H\cap M} & \xleftarrow{s_{MN}} & \frac{H}{H\cap N} & \xleftarrow{s_N} & H\cong\lim_{P\in\mathcal{N}(G)}\frac{H}{H\cap P}\\
{\scriptstyle f_M}\downarrow & & {\scriptstyle f_N}\downarrow & & \downarrow{\scriptstyle \lim_{P\in\mathcal{N}(N)} f_P}\\
\frac{HM}{M} & \xleftarrow{r_{MN}} & \frac{HN}{N} & \xleftarrow{r_N} & H=\lim_{P\in\mathcal{N}(G)}\frac{HP}{P}\\
{\scriptstyle \mathrm{inc}}\downarrow & & {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_H}\\
\frac{\overline{HM}}{M} & \xleftarrow[q_{MN}]{} & \frac{\overline{HN}}{N} & \xleftarrow[q_N]{} & H=\lim_{P\in\mathcal{N}(G)}\frac{\overline{HP}}{P}\\
{\scriptstyle \mathrm{inc}}\downarrow & & {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}}\\
\frac{G}{M} & \xleftarrow[p_{MN}]{} & \frac{G}{N} & \xleftarrow[p_N]{} & G=\lim_{P\in\mathcal{N}(G)}\frac{G}{P}.
\end{array}$$

We note, in particular, that $q_N(H)=\frac{HN}{N}$. For the Lie algebras $\mathfrak{L}(G)$ we write $\mathfrak{g}$, etc.

Lemma A. *Assume H to be connected. For each $N\in\mathcal{N}(G)$, the quotient $H/(H\cap N)$ is a Lie group with Lie algebra $\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{n})$.*

Proof. For $N\in G$ the morphism $f\colon H\to G/N$, $f(h)=hN$ has kernel $H\cap N$. The Lie group G/N has an identity neighborhood V in which $\{1\}$ is the only subgroup. Then $U\stackrel{\text{def}}{=}f^{-1}(V)$ is an identity neighborhood of H in which every subgroup is contained in $H\cap N$. Since H is assumed to be connected, this follows from Proposition 3.27 and Lemmas 3.23 and 3.24, Since $\lim\mathcal{N}(H)=1$, there is a $P\in\mathcal{N}(H)$ such that $P\subseteq U$, and thus $P\subseteq H\cap N$. Then $H/(H\cap N)\cong(H/P)/((H\cap N)/P)$ is a Lie group as a quotient of a Lie group. (Compare the proof of 3.29(ii).)
We have $\mathfrak{L}(H\cap N)=\mathfrak{h}\cap\mathfrak{n}$ since $\mathfrak{L}$ preserves limits, hence intersections (cf. Theorem 2.25(ii)). Let $q\colon H\to H/(H\cap N)$ be the quotient morphism. We claim that $\mathfrak{L}(q)\colon\mathfrak{L}(H)\to\mathfrak{L}(H/(H\cap N))$

is surjective; once this is shown we know $\mathfrak{L}(H/(H\cap N))\cong\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{n})$ and this will complete the proof of Lemma A.

So let us prove in general the following

Fact. *Let $q\colon H\to K$ be a quotient morphism of topological groups and assume that H is a connected pro-Lie group. Then $\mathfrak{L}(q)\colon\mathfrak{L}(H)\to\mathfrak{L}(K)$ is surjective.*

Indeed, for a proof, let $X\in\mathfrak{L}(K)$ and let $\beta\colon\mathbb{R}\to K$ be defined by $\beta(r)=\exp_K r{\cdot}X$. Set $P=\{(h,r)\in H\times\mathbb{R}: q(h)=\beta(r)\}$. Then P is a pull-back of pro-Lie groups and thus is a pro-Lie group such that $\alpha\colon P\to\mathbb{R}$, $\alpha(h,r)=r$ is a quotient. If we can show that this morphism induces a surjective morphism $\mathfrak{L}(\alpha)\colon\mathfrak{L}(P)\to\mathfrak{L}(\mathbb{R})=\mathbb{R}$ then X is shown to be in the image of $\mathfrak{L}(q)$ and the claim will be proved. (These matters will be discussed in greated detain in 4.16–4.18 below.)

By Lemmas 3.22 through 3.24 applied to P, we know that $P_0=\overline{\langle\exp_K\mathfrak{L}(P)\rangle}$. Thus $\langle\exp_{\mathbb{R}}\mathfrak{L}(\alpha)(\mathfrak{L}(P))\rangle == \alpha(\langle\exp_P\mathfrak{L}(P)\rangle)$ is dense in $\alpha(P_0)\subseteq\mathbb{R}$. As a connected subgroup of $\mathbb{R}$, the image $\alpha(P_0)$ is either $\{0\}$ or $\mathbb{R}$; in the second case we are done. In the first we obtain a quotient morphism $P/P_0\to\mathbb{R}$. By 3.30 and 3.31 below, which will be proved with the information we have provided up to this point, we know that every identity neighborhood U of P/P_0 contains an open closed normal subgroup. Thus $\mathbb{R}$. as a quotient of P/P_0, would have to have arbitrarily small open subgroups which is not the case. Thus $\alpha(P_0)=\mathbb{R}$ and the proof of the fact and then of Lamma A is complete. □

For the next step we accept the following facts: An arcwise connected subgroup of a Lie group is an analytic subgroup by the Theorem of Yamabe-Gotô (see [70]). If A is an analytic subgroup of a Lie group G, then there is a unique connected Lie group topology on A, possibly finer than the topology induced from G, making it into a Lie group $A_{\rm Lie}$ such that the inclusion map $A\to G$ induces an injective morphism of topological groups $f\colon A_{\rm Lie}\to G$, where $\mathfrak{L}(f)\colon\mathfrak{L}(A_{\rm Lie})\to\mathfrak{L}(G)$ is an injection of Lie algebras and $\operatorname{im}\mathfrak{L}(f)=\mathfrak{L}(A)$. See for instance [102], Theorem 5.52. The topology of $A_{\rm Lie}$ is the *arc component topology* of A (see [102], A4.1ff.).

Lemma B. *Assume that H is a* connected *closed subgroup of the pro-Lie group G. Then for each $N\in\mathcal{N}(G)$, the group $\frac{HN}{N}$ is an analytic subgroup of the Lie group G/N, and its Lie algebra is $\frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}$. The Lie group $\left(\frac{HN}{N}\right)_{\rm Lie}$ is isomorphic to $\frac{H}{H\cap N}$; indeed it is the image of $\frac{H}{H\cap N}$ under the standard bijective morphism $f_N\colon\frac{H}{H\cap N}\to\frac{HN}{N}$, $f_N(h(H\cap N))=hN$, given the topology making f_N a homeomorphism.*

Proof. Let us write $L=H/(H\cap N)$, whence we may write $\mathfrak{l}=\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{n})$ by Lemma A. Set $\mathfrak{a}\stackrel{\rm def}{=}\mathfrak{L}(f_N)(\mathfrak{l})$ and $A\stackrel{\rm def}{=}HN/N$. By Lemma A, L is a Lie group, and since H is connected, it is connected. Then $L=\langle\exp_L(\mathfrak{l})\rangle$ and we have

$$A=f_N(\langle\exp_L(\mathfrak{l})\rangle)=\langle f_N(\exp_L(\mathfrak{l}))\rangle=\langle\exp_G\mathfrak{L}(f_N)(\mathfrak{l})\rangle=\langle\exp_G\mathfrak{a}\rangle.$$

This means that A is the unique analytic subgroup of G generated by the (closed) subalgebra $\mathfrak{a}$ of $\mathfrak{L}(G/N)$. By [102], Theorem 5.52(iii), $\mathfrak{a}=\mathfrak{L}(HN/N)$. □

We recall that $\mathfrak{f}_N\colon\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{n})\to(\mathfrak{h}+\mathfrak{n})/\mathfrak{n}$ is an isomorphism of pro-Lie algebras (more generally, cf. Theorem A2.12(c).).

We abbreviate $\mathfrak{L}(\overline{HN})$ by $\mathfrak{h}_N$ and note $\mathfrak{L}(\overline{HN/N}) = \mathfrak{h}_N/\mathfrak{n}$ (see Corollary 4.21(i)). Our passing to the Lie Algebras in our big commutative diagram yields

$$\begin{array}{ccccc}
\frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{m}} & \xleftarrow{\mathfrak{s}_{MN}} & \frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{n}} & \xleftarrow{\mathfrak{s}_N} & \mathfrak{h} \cong \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}}{\mathfrak{h}\cap\mathfrak{p}} \\
{\scriptstyle \mathfrak{f}_M}\big\downarrow & & {\scriptstyle \mathfrak{f}_N}\big\downarrow & & \big\downarrow{\scriptstyle \lim_{P\in\mathcal{N}(G)} \mathfrak{f}_P} \\
\frac{\mathfrak{h}+\mathfrak{m}}{\mathfrak{m}} & \xleftarrow[\mathfrak{r}_{MN}]{} & \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} & \xleftarrow{\mathfrak{r}_N} & \mathfrak{h} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}+\mathfrak{p}}{\mathfrak{p}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathfrak{h}}} \\
\frac{\mathfrak{h}_M}{\mathfrak{m}} & \xleftarrow[\mathfrak{q}_{MN}]{} & \frac{\mathfrak{h}_N}{\mathfrak{n}} & \xleftarrow[\mathfrak{q}_N]{} & \mathfrak{h} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{h}_P}{\mathfrak{p}} \\
{\scriptstyle \mathrm{inc}}\big\downarrow & & {\scriptstyle \mathrm{inc}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inc}} \\
\frac{\mathfrak{g}}{\mathfrak{m}} & \xleftarrow[\mathfrak{p}_{MN}]{} & \frac{\mathfrak{g}}{\mathfrak{n}} & \xleftarrow[\mathfrak{p}_N]{} & \mathfrak{g} = \lim_{P\in\mathcal{N}(G)} \frac{\mathfrak{g}}{\mathfrak{p}}.
\end{array}$$

Again we observe that $\mathfrak{q}_N(\mathfrak{h}) = \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}$.

Lemma C. *For each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ contained in M such that $\frac{\mathfrak{h}_N}{\mathfrak{n}} = \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}$.*

Proof. Applying Lemma 3.18= Theorem A2.12 we find that for each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ such that $\mathfrak{q}_{MN}(\frac{\mathfrak{h}_N}{\mathfrak{n}}) \subseteq \mathfrak{q}_M(\mathfrak{h}) = \frac{\mathfrak{h}+\mathfrak{m}}{\mathfrak{m}}$. Let $X + \mathfrak{n} \in \frac{\mathfrak{h}_N}{\mathfrak{n}}$. Then we find an element $Y \in \mathfrak{h}$ such that $X \in X + \mathfrak{m} = \mathfrak{p}_{MN}(X + \mathfrak{n}) = \mathfrak{q}_{MN}(X + \mathfrak{n}) = \mathfrak{q}_M(Y) = Y + \mathfrak{m}$. Thus there is a $U \in \mathfrak{m}$ such that $X - Y = U$. Now $U + \mathfrak{n} \subseteq \mathfrak{h}_N \cap \mathfrak{m}$ since $\mathfrak{h} + \mathfrak{n} \subseteq \mathfrak{h}_N$. So $X + \mathfrak{n} = Y + U + \mathfrak{n} \in \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}$. This implies the claim. □

Now we let M^* be that subgroup of $\overline{HN} \cap M \subseteq M$ containing N for which $\frac{M^*}{N} = \left(\frac{\overline{HM}\cap M}{N}\right)_0$ in the Lie group G/N. Then M^* is normal in $\overline{HN}$.
As a consequence of Lemma 2.5, we get

Lemma D. *For each $M \in \mathcal{N}(G)$ there is an $N = N_M \in \mathcal{N}(G)$ contained in M such that*

$$\frac{\overline{HN}}{N} = \frac{HN}{N} \cdot \left(\frac{\overline{HN}\cap M}{N}\right)_0 = \frac{HN}{N} \cdot \frac{M^*}{N} = \frac{HM^*}{N}.$$

Proof. We have $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N/\mathfrak{n}\rangle$, further $\frac{HN}{N} = \langle \exp_{G/N} \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}\rangle$ by Lemma C. Finally

$$\frac{\mathfrak{h}_N}{\mathfrak{n}} \cap \frac{\mathfrak{m}}{\mathfrak{n}} = \ker \mathfrak{q}_{MN} \text{ and } \frac{\overline{HN}}{N} \cap \frac{M}{N} = \ker q_{MN},$$

whence $\left(\frac{\overline{HN}\cap M}{N}\right)_0 = \langle \exp_{G/N}(\frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}})\rangle$. Therefore, $\frac{\overline{HN}}{N} = \langle \exp_{G/N} \mathfrak{h}_N/\mathfrak{n}\rangle =$
$\langle \exp_{G/N} \left(\frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}} + \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}\right) = \langle \exp_{G/N} \frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}\rangle\langle \exp_{G/N} \frac{\mathfrak{h}_N\cap\mathfrak{m}}{\mathfrak{n}}\rangle = \frac{HN}{N} \cdot \left(\frac{\overline{HN}\cap M}{N}\right)_0 = \frac{HN}{N} \cdot \frac{M^*}{N} = \frac{HM^*}{N}$.
This completes the proof. □

Main Lemma E *Let H be a closed connected subgroup of the pro-Lie group G. Then for each $M \in \mathcal{N}(G)$ there is an $N_M \in \mathcal{N}(G)$ such that $N_M \subseteq M$ and that, for the subgroup $M^* \leq \overline{HN_M} \cap M \subseteq \overline{HN_M}$ such that $\frac{M^*}{N} = \left(\frac{\overline{HN_M}\cap M}{N}\right)_0$ the following statements are true:*

(i) $N_M \subseteq M^* \subseteq M$ *and* $\frac{\overline{HN_M}}{M^*} = \frac{\overline{HM^*}}{M^*}$ *is a Lie group.*

(ii) *The analytic subgroup* $\frac{HM^*}{M^*}$ *of the Lie group* $\frac{\overline{HM^*}}{M^*}$ *is closed.*

(iii) *The natural morphism* $\frac{H}{H\cap M^*} \to \frac{HM^*}{M^*}$ *is an isomorphism of Lie groups.*

Proof. We choose $N_M \in \mathcal{N}(G)$ as in Lemma D.

Proof of (i). Trivially, $N_M \subseteq \overline{HN_M}$ and by the choice of N_M we have $N_M \subseteq M$. Hence $N_M \subseteq \overline{HN_M} \cap M$. The inclusion $M^* \subseteq M$ is trivial. We note $\overline{HM^*} \subseteq \overline{HN_M} \subseteq \overline{HM^*}$. Since $N_M \in \mathcal{N}(G)$, the quotient G/N_M is a Lie group and thus the quotient $\frac{\overline{HN_M}}{M^*} \cong \frac{\overline{HN_M}/N_M}{M^*/N_M}$ is a Lie group as a quotient of a Lie group.

Proof of (ii). The quotient group $\frac{HN_M}{N_M}$ is an analytic subgroup of the Lie group G/N_M by Lemma B. The group HM^*/M^* is a continuous image of this arcwise connected group under the morphism $hN_M \mapsto hM^*$ and therefore, as an arcwise connected group, an analytic subgroup. By Lemma D we have $\frac{\overline{HN_M}}{N_M} = \frac{HM^*}{N_M}$. Passing to the quotient modulo M^*/N_M on both sides yields $\frac{\overline{HN_M}/N_M}{M^*/N_M} = \frac{HM^*/N_M}{M^*/N_M}$, which is equivalent to $\frac{\overline{HN_M}}{M^*} = \frac{HM^*}{M^*}$; this implies that $\frac{HM^*}{M^*}$ is closed and is, therefore, a Lie group.

Proof of (iii). The natural bijective morphism of topological groups

$$h(H \cap M^*) \mapsto hM^* : \frac{H}{H \cap M^*} \to \frac{HM^*}{M^*}$$

between two connected Lie groups is an isomorphism by the Open Mapping Theorem for Locally Compact Groups. □

The situation is illustrated by the following diagram:

$$\begin{array}{ccccccc}
\frac{HM}{M} & \longleftarrow & \frac{HM^*}{M^*} & \longleftarrow & \frac{HN_M}{N_M} & \longleftarrow & H = \lim_{P\in\mathcal{N}(G)} \frac{HP}{P} \\
\text{inc}\Big\downarrow & & =\Big\downarrow & & \Big\downarrow\text{inc} & & \Big\downarrow \text{id}_H \\
\frac{\overline{HM}}{M} & \longleftarrow & \frac{\overline{HM^*}}{M^*} & \longleftarrow & \frac{\overline{HN_M}}{N_M} & \xleftarrow{q_N} & H = \lim_{P\in\mathcal{N}(G)} \frac{\overline{HP}}{P} \\
\text{inc}\Big\downarrow & & & & \Big\downarrow\text{inc} & & \Big\downarrow \text{inc} \\
\frac{G}{M} & \underset{p_{MN}}{\longleftarrow} & \frac{G}{N_M} & \underset{=}{\longleftarrow} & \frac{G}{N_M} & \underset{p_N}{\longleftarrow} & G = \lim_{P\in\mathcal{N}(G)} \frac{G}{P}.
\end{array}$$

We observe that M^* need not be a member of $\mathcal{N}(G)$. If H is a normal subgroup, then $\overline{HN_M} \cap M$ is normal in G, and since $\frac{M^*}{N_M} = \left(\frac{\overline{HN_M}\cap M}{N_M}\right)_0$ is characteristic in $\frac{G}{N_M}$, the group M^* is invariant under all automorphisms of G leaving N_M invariant, and so certainly under all inner automorphisms of G; and thus M^* is a member of $\mathcal{N}(G)$. Therefore we have the

Lemma F. *Let H be a closed connected normal subgroup of the pro-Lie group G. Then for each $M \in \mathcal{N}(G)$ there is an $M^* \in \mathcal{N}(G)$ such that $M^* \subseteq M$ and that the following statements are true:*

(i) *The analytic subgroup* $\frac{HM^*}{M^*}$ *of the Lie group* $\frac{G}{M^*}$ *is closed.*

(ii) $h(H \cap M^*) \mapsto hM^* : \frac{H}{H\cap M^*} \to \frac{HM^*}{M^*}$ *is an isomorphism of Lie groups.*

□

At this time we have proved the theorem for *connected* closed subgroups H. We now complete its proof by showing that the results Lemma E remain intact for an *almost* connected closed subgroup H of a pro-Lie group G.
Thus let H be a closed almost connected subgroup of the pro-Lie group G and let $M \in \mathcal{N}(G)$. We apply Lemma D to H_0 in place of H and find that there is a closed normal subgroup M^* of $\overline{H_0M}$ contained in M such that H_0M^*/M^* is a connected Lie group. In particular, H_0M^* is a closed subgroup of G. We must show that HM^*/M^* is a Lie group; the remainder of the theorem then follows. Now H/H_0 is assumed to be compact. The continuous morphism

$$hH_0 \mapsto hH_0M^* : H/H_0 \to HM^*/H_0M*$$

is surjective. Hence HM^*/H_0M^* is a compact group. Thus HM^*/M^* is a locally compact group as an extension of the Lie group H_0M^*/M^* by the compact group HM^*/H_0M^*. The morphism

$$h(H \cap N_M) \mapsto hM^*: H/(H \cap N_M) \to HM^*/M^*$$

is a surjective morphism from a σ-compact group onto a locally compact group and is therefore open. The group $H/(H \cap N_M)$ is a Lie group. Thus HM^*/M^* is a Lie group. This completes the proof of the theorem.

---

page 152, 3.29(iii)	replace by the following:

(iii) The set $I = \{j \in J : G_0/(G_0 \cap K_j) \to (G_0K_j)/K_j$ is an isomorphism of topological groups$\}$ is cofinal in J. For $j \in I$, the group G_0K_j is a Lie group and a closed subgroup of G/K_j, and $G_0 = \lim_{j\in J}(G_0K_j)/K_j$.

---

page 152, line 2 of 3.29(iv)	read: G_0/M into a Lie group.
page 152, lines -7,-6	replace "In particular, ... (iii)." by the following:

Theorem 3.28 shows that the set I is cofinal in J, and it establishes the other statements of (iii) as well.

---

page 152, lines -4	read: of the form $M = G_0 \cap \ker f_j$
page 152, lines -2,-1	replace by the following:

---

an injective morphism $G_0/M \to G_0K_j/K_j \to G/K_j \xrightarrow{F_j} G_j$ where $F_j: G/K_j \to G_j$ is the unique morphism such that $f_j = F_j \circ q_j$ for the quotient map $q_j: G \to G/K_j$.

---

page 154, line 2	replace "π'_j" by $\pi_{j'}$".
page 156, line 1	replace "$= \delta vN$" by $= \delta vM$".
page 156, line 5	replace "B/M" by "B/N".
page 156, line -12(not counting diagr.)	replace "4.3(b)" by "3.30(b)".

page 157, line -7(not counting box)	replace “Lemma 4.5” by “Lemma 3.32(i)”.
page 160 Example 3.38	add period at the end of the paragraph.
page 169, line 2	replace “morphism” by “monic”.
page 169, lines 4,5,6	replace by the following text:

---

then a has a unique factorisation $a = a' \circ \epsilon$. We mention in passing that categories in which every morphism ϕ has such a factorisation $\phi = \mu \circ \epsilon$ are said to have *epic-monic factorisation.*

---

page 169, line -12	replace “q_f” by “q_ϕ.
page 169, line -12	replace “q_f” by “q_ϕ.
page 171, line 15	read “Let G be a proto-Lie group”
page 174, lines 3, 5	replace “P” by “G”.
page 182, line 6	read “of nonzero Hausdorff topological”.
page 182, line 7	read “Then $E \overset{\text{def}}{=} \ldots$”.
page 182, line 1 of Exercise E4.1	add period at the end of the line .
page 182, line -8	read “Theorem 7.30(iv)]); in particular, the”.
page 183: line 1	read “ Assume that f is a quotient morphism and”.
page 183, 4.16(i)	insert before period: “and open”.
page 183	Exchange the order of 4.16 and 4.17.
page 188, line-10 not counting box	before “Also recall” insert the following:

A short sequence of topological groups $N \xrightarrow{e} G \xrightarrow{f} H$ is called *strict exact* if e is a strict morphism and $\operatorname{im} e = \ker f$. An exact sequence

$$1 \to N \xrightarrow{e} G \xrightarrow{f} H \to 1$$

is called *strict exact* if the sequence of the middle three terms is strict exact.

---

page 189, diagram (*)	the label of the right downarrow reads “$\mathfrak{L}(\nu_N)$”.
page 189, line -8	replace “Theorem 1.20(i)” by “Theorem 1.29(i)”.
page 190, line -5:	replace “$\mathfrak{L}(f)(X)$” by “$\mathfrak{L}(q)(X)$”.
page 193, lines 4–14	replace by the following:

(b) (H. Glöckner) By Theorem A2.12(b)of Appendix 2, the map $\mathfrak{L}(f)$ will be surjective if it has dense image. As a consequence of Corollary 4.21 (ii), the latter is the case if $L(\nu_N \circ f) \colon \mathfrak{L}(G) \to \mathfrak{L}(H/N)$ is surjective for each $N \in \mathcal{N}(H)$, where $\nu_N \colon H \to H/N$ is the quotient morphism. We may therefore assume that H is a Lie group. But then f factors through a surjective morphism $g \colon G/M \to H$ for some $M \in \mathcal{N}(G)$. Since G/M is an almost connected Lie group and hence is σ-compact, g is a quotient morphism by the Open Mapping Theorem for Locally Compact Groups. Hence $\mathfrak{L}(g)$ (and thus $\mathfrak{L}(f)$) is surjective.

---

page 198, line 16	read “. . .and locally compact, and G/M is locally compact.”.

page 199, line -13 — replace "$g_{J\setminus E_B}$" by "$g(B)_{J\setminus E_B}$".
page 202, line 7 — Replace the sentence starting "But $\mathbb{Z}$ fails to... by the following:

The topology on $\mathbb{Z}$ having as a subbasis for its closed sets the collection of cosets modulo nonsingleton subgroups of $\mathbb{Z}$ fails to be compact.

---

page 203, line 11 from below — replace "Lemma" by "Corollary".

---

page 204, line 7 — replace by the following:

If (ii*) implies the conclusion of 4.28, then so does (ii), and (ii*) is implied by

---

page 205, line -5 — add period at the end of the line.
page 206, line -17 — At the end of the line add "]".
page 207, line 12 — After "Then" insert "for each topological group H,"
page 209, line 7 — read "$N \in \mathcal{N}(G)$, then".
page 210, Exercise E4.8 — replace "[Hint. ...] and the subsequent paragraph by the following:

[Hint. A metric space is completely regular. The real valued functions on a completely regular space separate the points. The continuous image of a connected completely regular space C in $\mathbb{R}$ is an interval; the cardinality of an interval in $\mathbb{R}$ is 1 or that of the continuum and so card(C) is 1 or $\geq 2^{\aleph_0}$.]
There are countable connected Hausdorff spaces. (See for instance [52], pp. 352, 353.)

---

page 226, lines 3,4,5 — replace by the following:

$\dots G_1 \cap U$. Now let $M \in \mathcal{N}(G)$ be contained in U. Then $M \cap G_1 \subseteq U \cap G_1 = \{0\}$. Let $N \stackrel{\text{def}}{=} M^* \subseteq M$ be the subgroup attached to M by Theorem 3.28. Since G is abelian, N is normal. Then G/N is a Lie group and $(G_1 + N)/N$ is isomorphic to $G/(G_1 \cap N) \cong G_1$ by Theorem 3.28. So $(G_1 + N)/N \dots$

---

page 226, lines 4,5,6 of 5.18 — Omit lines 4 and 5 and read $(*)$ as follows:

$$(*) \qquad F + \bigcap \mathcal{F} = \bigcap_{H \in \mathcal{F}} (F + H).$$

---

page 235, line 9	read: prodiscrete abelian group
page 236, line-8	read: $\cdots \cong \mathbb{R}^m \times \text{comp}(G) \times \mathbb{Z}^n$.
page 238, line 10	read: also Chapter 14, Example 14.15.) Therefore...
page 242, line-7	read: $\mathfrak{L}(G) \to \mathfrak{L}(G/D)$
page 242, line -5	add period at the end of the line.
page 249, line-7	Prior to line 7 from below insert a paragraph as follows:

We shall show later in Theorem 8.15 on p. 344 that a pro-Lie group is prosimply connected iff it is simply connected.

---

page 250, line-2,-1 above 6.3	replace "and the one ... $\pi_1(G)$." by the following:

---

and implies the one expressed by the vanishing of the fundamental group $\pi_1(G)$ while not being implied by it.

---

page 252, bottom diagram	southeast corner should read "$G_{\mathfrak{i}}$".
page 254, bottom diagram	northwest corner should read "$\mathfrak{L}(\Gamma(\mathfrak{g}))$".
page 260, line 2	replace "$\Gamma(h)$" by "$\Gamma(\mathfrak{h})$".
page 261, line -9	replace "$\Gamma(h)$" by "$\Gamma(\mathfrak{h})$".
page 261, line -6	after the period insert:

Define $\widetilde{H} = \Gamma(\mathfrak{L}(H))$ (cf. Theorem 4.20).

---

page 262, top diagram	in the labels of all vertical arrows replace "p" by "π".
page 262, line 2 below top diagram	read "Let G be a prosimply".
page 263, line 8	Replace "$\widetilde{g}$" by "$\widehat{G}$".
page 264, line-1	replace the line by the following:

If (i) is satisfied, then H and K may be taken to be $\phi(G)$, and σ the corestriction of ϕ.

---

page 265, line 9	replace the line by the following:

If (i) is satisfied, then $\mathfrak{h}$ and $\mathfrak{k}$ may be taken to be $\Phi(\mathfrak{g})$, and ρ the corestriction of Φ.

---

page 271, line 5	replace "topological vector space" by 'continuous L-module".
page 272, line 9	read $x{\cdot}\omega = -\omega \circ \text{ad}\, x$
page 273, line 9	replace "$\widehat{V^\star}$ by "$\widehat{V}^\star$".
page 273, line 22	replace "subset" by "vector subspace".
page 273, line 23:	after "submodule", insert "of a continuous L-module" and replace "an L-module" by "a continuous L-module".

page 275, line 2	replace "topological" by "continuous".
page 276, line 19	insert comma after "equalizers".
page 276, line 2 of Definition 7.13	"natural transformation" for "cone".
page 280 7.20 and proof	replace by the following:

Proposition 7.20. (i) *Assume that E is a simple L-module. Then E agrees with E_0 only if it is zero or one-dimensional; if this is not the case, then E agrees with E_{eff}. In the latter case, if $0 \neq v \in E$, then $\langle v\rangle_L = E$.*
(ii) *If E is a semisimple locally finite-dimensional L-module, then E is the direct sum $\sum_{j\in J} E_j$ of a family of simple submodules E_j of E. Let $J_0 = \{j \in J : E_j = (E_j)_0\}$ and $J_1 = \{j \in J : E_j = (E_j)_{\text{eff}}\}$. Then*

(3) $$E_0 = \sum_{j\in J_0} E_j,$$

(4) $$E_{\text{eff}} = \sum_{j\in J_1} E_j.$$

In particular,

(5) $$E = E_0 \oplus E_{\text{eff}}.$$

Proof. (i) Let E be a simple L-module. Then E_0 is either $\{0\}$ or E. In the latter case, every vector subspace being a submodule, its dimension is either 0 or 1. In the former case, let $0 \neq v \in E$. Then $L{\cdot}v$ is a submodule. Since $E_0 = \{0\}$ it is nonzero. Hence $L{\cdot}v = E$ and $E = E_{\text{eff}}$.
(ii) From 7.16 we deduce the direct sum representation $E = \sum_{j\in J} E_j$. Define $F_0 = \sum_{j\in J_0} E_j$, and $F_1 = \sum_{j\in J_1} E_j$. Let $0 \neq v_j \in E_j$. Then

$$L{\cdot}x_j = \begin{cases} \{0\} & \text{if } j \in J_0, \\ E_j & \text{if } j \in J_1 \end{cases}$$

by (i) above and thus $F_0 \subseteq E_0$, $F_1 \subseteq E_{\text{eff}}$. Since $J = J_0 \cup J_1$ and $J_0 \cap J_1 = \emptyset$ we have $E = F_0 \oplus F_1$. Now let $v \in E_0$. Then $0 = x{\cdot}v = \sum_{j\in J} x{\cdot}v_j$ for all $x \in L$, and thus $x{\cdot}v_j = 0$ for all $x \in L$ and $j \in J$. Hence $v_j \neq 0$ implies $v_j \in E_{j_0}$, and that implies $v \in F_0$ Therefore $F_0 = E_0$.
But $F_0 \oplus F_1 = E$ and $F_1 \subseteq E_{\text{eff}}$ imply $E_{\text{eff}} = (F_0 \cap E_{\text{eff}}) \oplus F_1$. Now let $v = \sum_{j\in J_0} v_j \in F_0 \cap E_{\text{eff}}$. Then $v = \sum_{m=1}^n x_m{\cdot}w_m$ with $x_m \in L$, $w_m \in E$. Now $w_m = \sum_{j\in J} w_{mj}$ with $w_{mj} \in E_j$, and

$$v = \sum_{\substack{m=1,\dots,n\\ j\in J}} x_m{\cdot}w_{mj} = \sum_{\substack{m=1,\dots,n\\ j\in J_1}} x_m{\cdot}w_{mj} \in F_1.$$

But then $v \in F_0 \cap F_1 = \{0\}$, and therefore $v = 0$. It follows that $F_0 \cap E_{\text{eff}} = \{0\}$, and thus that $F_1 = E_{\text{eff}}$. □

---

page 281 lines 3 of Theorem 7.22	remove “$1 <$”.
page 281 lines -6, -5	read:

7.20(ii), this module is a direct sum of simple finite dimensional modules. Thus $\overline{V_{\text{eff}}}$, by duality, is a product of simple finite dimensional modules.

---

page 286, line 17	replace “card $\mathfrak{g}$” by “card $\overline{g}''$.
page 286, line -11	replace γ” by “$\overline{\gamma}$”.
page 286, line -1	replace “$= \mathfrak{g}^{(\beta)}$” by“$\supseteq \mathfrak{g}^{(\beta)}$”.
page 287, line 11	replace “card $\mathfrak{g}$” by “card $\overline{g}''$.
page 287, line 14	replace “$= \mathfrak{g}^{(\beta)}$” by“$\supseteq \mathfrak{g}^{(\beta)}$”.
page 287, line 1 of Definition 7.39	replace “δ” by “$\overline{\delta}$.
page 289, lines 3,4,5 of 7.43	omit lines 4 and 5 and read $(*)$ as follows:

$$(*) \qquad F + \bigcap \mathcal{F} = \bigcap_{H \in \mathcal{F}} (F + H).$$

---

page 289, line 3 of 7.44	read $= (\mathfrak{g}^{((\alpha))} + \mathfrak{j})/\mathfrak{j}$.
page 292, lines 1 and 6	replace “solvable” by “transfinitely topologically solvable”.
page 292, line 8	replace “$\mathfrak{r}(\mathfrak{g}) + \mathfrak{i}$” by “$\overline{\mathfrak{r}(\mathfrak{g}) + \mathfrak{i}}$”.
page 293, line 3 of Definition 7.49	replace “space” by “spaces”.
page 297, line 1	delete “finite-dimensional”.
page 297, line 13	read $= (\mathfrak{g}^{[[\alpha]]} + \mathfrak{j})/\mathfrak{j}$.
page 300, line 5 of 7.59	replace “$\mathfrak{g}{\cdot}E \subseteq F$” by “$\mathfrak{g}{\cdot}x \subseteq F$”.
page 301, line 12	replace “topological $\mathfrak{g}$-module” by “continuous $\mathfrak{g}$-module”.
page 302, line -8	replace “almost” by “locally”.
page 303, line -2 above 7.66	replace “7.65” by “7.64”.
page 305, line -3	delete “$x \in V$ and $\omega \in E$.”
page 305, line -1	before the period insert “for $x \in V$ and $\omega \in E$.”
page 308, line -7	replace “derivation” by “linear self-map”.
page 314, line 1 of 7.80	replace “Let” by “Assume that iis an ideal and let”
page 314, line 2 of 7.80	read “$\mathfrak{g} - module$.
page 314, line 3 of 7.80	replace “the annihilators in $\mathfrak{g}$” by “are the annihilators of $\mathfrak{i}$, respectively, $\mathfrak{a}$ in $\mathfrak{g}_{\text{coad}}$”.
page 314, line 4 of 7.80	read “$\mathfrak{g}$-submodule”.
page 314, line 10 of 7.80	read “is a retraction”.
page 319, line -1	read “the adjoint action”.
page 342, line 2 of 8.13	replace “Lie radical” by “nilradical”.
page 342, line 3 of 8.13, ll.-6,-3	replace “reductive” by “coreductive”.

page 343, line 10 of 8.14	replace “isomorphism” by “homeomorphism”.
page 354 lines 17ff.	displaylines (i)—(iv), replace by the following:

(i) G is prosimply connected;
(ii) each covering map $f\colon E \to G$ of topological spaces maps every connected component of E homeomorphically onto G.
(iii) The “universal covering morphism” $\pi_G\colon \widetilde{G} \to G$ is bijective.

These statements imply

(iv) G is connected and every loop based at the identity can be homotopically contracted, that is, $\pi_1(G) = 0$,

but are not implied by it; in fact there are compact connected abelian groups G with $\pi_1(G) = 0$, while no compact abelian group is simply connected (see [102], Theorems 8.62, 9.29, and also. Example 6.2 above.)

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page 355, line -7	replace “]. In” by “]). In”.
page 357, line 3	delete “dual with its”.
page 357, line 12	read “by a power series”.
page 357, line 14	replace “$x \in F$” by “$\omega \in F$”.
page 358, line 4 of 9.1	read “filter basis $\mathcal{F}$ of ψ-invariant closed vector subspaces”.
page 360 lines 7,8	omit “and... $\mathfrak{L}(H)$”.
page 360 line 9	Between 9.4 and 9.5 insert paragraph:

By Theorem A2.12(a), the image $\mathfrak{L}(f)\big(\mathfrak{L}(G)\big)$ is a closed Lie subalgebra of $\mathfrak{L}(G)$.

---

page 360 line 11	replace “a strict” by “an analytic”
page 360 line 12	replace “strict” by “analytic”
page 360 line 12, 13	replace “the paragraph... 4.20 ” by “Definition 9.4”
page 360 line 14	delete “and $\mathfrak{L}(f)(\mathfrak{L}(C))$ is closed”.
page 362 lines 5,6,-15,-14,-13,-12	replace formula letter “H” by “A”.
page 376, line-5 (without headline)	delete “connected”.
page 377, line 9 of 9.25	After “(iii)” insert “*Under the hypotheses of* (ii),
page 385, line -12	replace “of T” by “$of\,\Delta$”.
page 385, line -6	After display line insert the sentence:

That is, δ is open onto its image and $\operatorname{im}\delta = \ker\mu$.

---

page 387, line -13	in the display formula, read:

$$\beta : \frac{\Delta \times \widetilde{G}}{D} \to G_1, \ldots$$

page 389, line 6	replace “free group” by “free abelian group”.
page 389, diagram (∗), l.-9	replace ‘δ” by “θ”.
page 391, line-8	read: (... is dense).
page 397, line-19	read: ...group L/N_0 and...
page 397, line-11	read:... $\to \mathfrak{L}(N_j/(N_j)_0) \to \dots$
page 402, line-18	at the beginning of the line, add “{”.
page 402, line-3	delete “finite dimensional”.
page 403, line 17	remove one period.
page 405 lines –20 through –14	replace by the following text

Fact. *A compact group is divisible iff it is connected.*

(See [102, Theorem 9.35]. For abelian compact groups see [102, Theorem 8.4]). The additive group of the field

page 406, line 2 of 9.54	replace “$G = CG$” by “$L = GC$”.
page 406, lines -8,-7	Replace “By Theorem 1.34(iv)... as topological groups,” by We know that $C/(C \cap N) \cong CN/N$ as groups,...
page 407, line -17	replace “Then $G = CG$” by “Then $\overline{G} = CG$”.
page 413, line -13	Delete one period.
page 414 line -12	replace displayline by the following:

$$G \stackrel{\text{def}}{=} \{(h, \omega(h)) : h \in H\} \subseteq \overline{G} \stackrel{\text{def}}{=} H \times G_1$$

page 416, line 12	delete “but that” .
page 416, line -9	replace “a strict” by “an analytic”.
page 419, line -11	replace “Lie groups” by “pro-Lie groups”.
page 420, lines 14, 16	replace “$\{0\}$” by “$\{1\}$”.
page 421, line 10	read “is a terminating abelian sequence”.
page 421, lines -11, -9	replace “$\{0\}$” by “$\{1\}$”.
page 421, line -7	replace “The sequence” by “The sequence $(N_\alpha)_{\alpha\le\rho}$”.
page 422, line -17, -5	replace “card $\mathfrak{g}$” by “card $\overline{G}$”.
page 422, line -8	replace “$\{0\}$” by “$\{1\}$”.
page 423, line 3	replace “$G^{[\infty]} = \{0\}$” by “$G^{[[\infty]]} = \{1\}$”.
page 423, line 6	replace “$\{0\}$” by “$\{1\}$”.
page 423, line 4	replace “δ” by “$\overline{\delta}$”.
page 434, line 1	replace “$\mathbb{N}$” by “$\mathbb{Z}$” (three times).
page 435, line -8, -7	delete “commutative”.
page 438, line -17	read “and thus $G/Z(G)$ is its own”.
page 443, line 1 of E10.7	add period at the end of line.
page 445, line 13	delete “transfinitely”.

page 447, lines -5, -4:	Omit the sentence "From an observation…closed in $\mathfrak{g}$".
page 453, line -1	replace "1 otherwise" by "0 otherwise".
page 455, line -1	read $(2,\mathbb{R})$.
page 456, line 2 of P10.2	replace "any" by "every".
page 456, line -4	replace "group $\mathfrak{L}(G)$" by "group G".
page 456, line -3	delete one parenthesis ")" after "$A(\mathfrak{s}, G)$".
page 457, line -9	replace "and G" by "and C".
page 457, line -8	replace "and $\mathrm{comp}(G)$" by "and $\mathrm{comp}(C)$".
page 458, line -1 above "Postscript"	add period at the end of the line.
page 461, line 1 of 11.2	read "Every pro-Lie algebra $\mathfrak{g}$ is".
page 462, line -6	read "There is a subgroup A of G such that".
page 464, line -20	replace "agree on D" by "agree on $D = N \cap H$".
page 466, line 2 of 11.8	replace "$\langle \exp_G \mathfrak{g}\rangle$ by "$\langle \exp_G \mathfrak{s} >\rangle$".
page 468, line -4	add period at the end of the line.
page 470, line 9	read $\exp'_A\colon \mathfrak{L}(A) \to A_a$.
page 473, line -5	read "an example of a connected pro-Lie group H"
page 475, lines 2,3	replace by the following:

---

so, since N is finite dimensional, there is a $P \in \mathcal{N}(G)$, such that $N \cap P = \{1\}$. Now let $M = P^* \subseteq P$ be the member of $\mathcal{N}(G)$ attached to P according to Theorem 3.28. Then by Theorem 3.28 $n \mapsto nM : N \to NM/M$ is an isomorphism, and…

---

page 484, line 5 of 11.27	add period at the end of the line.
page 489, line 2 of 11.31	delete "there are".
page 491, line 7	replace "Z^J" by "$\mathbb{Z}^J$" and "$A(\mathfrak{s}; G)$" by "$A(\mathfrak{s}, G)$".
page 495, Section (iv) of 12.4:	omit.
page 496, line 7	omit, move $\square$ up.
page 496, line 10	omit
page 496, line 11	replace "(c)" by "(b)".
page 496, line 18	in "Hint" omit all of (b) and rename "(c)" as "(b)".
page 497, line 7	Add: "Recall Definition 7.15(iii) of a reductive profinite dimensiona L-modules".
page 497, line2 of 12.6	replace "*semisimple L-module*" by "*reductive L-module*.
page 497, line 4 of Definition 12.7	preceding the period add the phrase

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and being such that the maps $v \mapsto g{\cdot}v : E \to E$ are linear for all $g \in G$

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page 497, line -4	add period at the end of the line.
page 498, lines 1,2	replace by the following:

The groups we consider in this book frequently fail to be locally compact. However, all weakly complete topological vector spaces are Baire spaces: Indeed by Corollary A2.9 on p. 638, weakly complete topological vector space is isomorphic as topological vector space to

$\mathbb{R}^J$ for some set J. But by [26], §5, Exercise 17, a product of completely metrizable spaces is a Baire space. In particular, by Proposition 7.9, all profinite-dimensional L-modules are Baire spaces. (See also Oxtoby, J. C., Cartesian products of Baire spaces. Fundamenta Math. **49** (1961), 157–166, Theorem 6. From this reference and Corollary 12.86 below it will follow that every connected pro-Lie group is a Baire space.)

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page 500, line 4 of Definition 12.10	add period after “algebra”.
page 508, line 3 of Lemma 12.21	replace “of G” by “of $\mathfrak{g}$”.
page 508, line 6 of Lemma 12.21	read “If $\mathfrak{t}$ is a compactly”.
page 514, line -3	add a period at the end of the line.
page 518 line 1 of the pf. of 12.37	read: From the preceding Theorem 12.36
page 519, line -12	replace “Ad_G” by “Ad”.
page 521, line 15	replace semicolon by comma.
page 525, line 8 of the proof of 12.53	replace “12. By” by “12.27(i). By”.
page 532, line 2	delete the first occurence of “connected”.
page 535: line -16	add period at the end of the line.
page 542, line 5 of Lemma 12.73	replace the word “implements” by “is surjective and implements”.
page 544 line -1	add period at the end of the line.
page 547 line 1 of 12.77	read: “of a connected pro-Lie group is”
page 550, line 2	read “*N a closed normal subgroup, and*”.
page 550, line 13	replace “$\mathfrak{p}$” by “p”.
page 550, line -6	replace “$C * \exp\ldots$ by $C^*exp\ldots$.
page 551, lines 7,6 above box	replace “$\mu*$” by “μ^*”.
page 551, line-11	replace “*group C*” by “*subgroup C*”.
page 556, line-7	replace “lemma” by “theorem”.
page 567, line 2 of 13.1(ii)	read “induce isomorphisms $\mathfrak{L}(\pi_{N\rtimes_\alpha G/N}), \mathfrak{L}(\mu)$”.
page 567, line 2 of 13.1(iii)	remove “, $\phi\colon \widetilde{G/N} \to G$,”.
page 569, line 3	At the end of the line insert “It helps to consider the diagram”.
page 569, line-3,-2	replace “12.89” by “12.88” (4 times).
page 570, line 13	At the beginning of the line, insert: “If G is connected, then” (N/N_0 is locally…)
page 571, line 01	read “…Theorem 9.44, if G is connected, the factor group $G/N_0\ldots$”
page 571, line 10	delete “and is closed”.
page 571, line 11	after “is normal” add “in $G_0 = G$”.
page 571, line-12	After the period add the following sentences:

A finite dimensional vector subspace of a (Hausdorff) topological vector space is locally compact and therefore closed (see Proposition A2.2). Thus every complemented ideal is automatically closed.

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page 571, line-5	read well-complemented

page 573, line-2	delete the second $\mathfrak{f}$.
page 574, line 3	replace "∞ then" by "∞. Then".
page 576, line 4	read $\mathfrak{n}(\mathfrak{g})/\mathfrak{z}(\mathfrak{g}) = \mathfrak{n}(\mathfrak{g}/\mathfrak{z}(\mathfrak{g}))$.
page 577, line-9	read "*cofinite-dimensional closed ideals*".
page 579, line -5	read "Let G be a connected pro-Lie group..."
page 579 line -4	replace "G_0" by "G" (twice).
page 580, line 4	replace "$\mathfrak{g}/\mathfrak{n}(\mathfrak{g})$" by "$\mathfrak{n}(g)/\mathfrak{z}(\mathfrak{g})$".
page 581, line 1 of 13.20	delete "pro-Lie".
page 583, line-12	read "locally compact group is".
page 583, line-11	replace "4.2" by "13.22".
page 585, line 20	in "topological vector space" cancel "vector".
page 586, line 8	replace "pro-Lie" by "Lie".
page 624, line -8	after = read $[(\mathrm{ad}\,x)y, z] + [y, (\mathrm{ad}\,x)z]$.
page 626, line 6	replace "*are linear*" by "*are* $\mathbb{Q}$*-linear*".
page 630, line 16	replace "in $\mathcal{U}$" by "is $\mathcal{U}$".
page 634, line 4 of Lemma A2.6	add period at the end of the line.
page 639, line 9	delete "a" before "vector spaces".
page 639, line 5 of A2.10	replace "$\ldots(\mathfrak{g}_k) \subseteq \ldots$" by "$\ldots(\mathfrak{g}_{k_j}) \subseteq \ldots$" .
page 641, lines-3,-2,-1 of A2.13	omit lines -2 and -1 and read (∗) as follows:

$$(*) \qquad F + \bigcap \mathcal{F} = \bigcap_{H\in\mathcal{F}} (F + H).$$

page 642, lines 13-17	omit.
page 652, line -2	replace "vector spaces" by "vector space".
page 654, line -9	replace "into isotypic" by "into the isotypic".
page 655, line -9	replace "[19, p. 75]" by "[19, p. 75])".
page 657, entry [10]	replace by the following:

[10] Borel, A., *Limites projectives de groupes de Lie*, C.R. Acad. Sci. Paris **230** (1950), 1127–1128.

[We owe the new reference [10] to a hint by HELGE GLÖCKNER.]

page 659 entry Goto, M.	replace "Gôto" by "Gotô"
page 664 entries [160], [161]	replace by the following:

[160] Onishchik, A. L., and E. B. Vinberg, eds., Lie groups and Lie algebras II, III Encyclopaedia Math. Sci. **21**, resp., **41** Springer-Verlag, Berlin 2000, resp., 1994
[161] Oxtoby, J. C., Cartesian products of Baire spaces, Fundamenta Math. **49** (1961), 157–166.